Int. J. Solids Structures Vol. 20, No. 8, pp. 761–776, 1984
Printed in the U.S.A.

0020-7683/84 $3.00 + .00

DYNAMIC BUCKLING OF VISCOPLASTIC SPHERICAL SHELL

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(Received 4 March 1983; in revised form 2 September 1983)

Abstract—The solution of viscoplastic buckling of a complete thin spherical shell subjected to impulse pressure is given. The nonlinear flow law is assumed and the influence of elevated temperature on the magnitude of displacements, buckling mode and threshold impulse is discussed. The special cases of bucking modes are also considered. The numerical results are presented diagrammatically for a steel shell loaded by a radial impulse.

1. INTRODUCTION

The problem of dynamic inelastic buckling of shells has been investigated in many papers. The solutions for a spherical shell made of an elastic–plastic and rigid–plastic material with linear hardening are given by Jones and Ahn[1, 2]. More attention was devoted to obtain solutions for a cylindrical shell: Abrahamson and Goodier[3], Lindberg[4], Anderson and Lindberg[5], Vaughan and Florence[6] and others. All those solutions concern shells made of strain-rate insensitive material.

The viscosity effects of the material are taken into account in the papers by Perrone[7], Florence[8], Florence and Abrahamson[9], Wojewódzki[10, 11] in which the cylindrical shell was considered and only a radial component of the displacement was accounted for. Wojewódzki and Lewiński[12] considered an axisymmetrical buckling of a complete spherical shell and investigated the influence of the meridional displacement on the magnitude of radial displacement, buckling mode and critical impulse. In these solutions a linear law of viscoplastic flow was used.

The present paper aims at solving the problem of a general buckling mode of a complete thin spherical shell loaded uniformly by an impulse pressure. The nonlinear law of viscoplastic flow will be assumed and the influence of elevated temperature will also be shown.

2. BASIC EQUATIONS

The constitutive equations. The influence of work hardening, strain rate and temperature on the material response can be described by the following equations formulated by Perzyna and Wierzbicki[13, 14]:

$$\dot{\epsilon}_{ij}=\frac{\gamma(\vartheta)}{2}\langle\Phi(F)\rangle\frac{s_{ij}}{J_2^{1/2}},\ F=\frac{J_2^{1/2}}{k(\vartheta)}-1,\ \langle\Phi(F)\rangle=\begin{cases}\Phi(F) & \text{for}\quad F>0\\ 0 & \text{for}\quad F\leqslant 0,\end{cases} \tag{2.1}$$

where $\dot{\epsilon}_{ij}$ is the strain rate tensor (dots indicate time differentiation), $J_2=\frac{1}{2}s_{ij}s_{ij}$ denotes the second invariant of the stress deviator s_{ij}, $i,j=1,2,3$, $k(\vartheta)=\sigma_0(\vartheta)/3^{1/2}$, $\sigma_0(\vartheta)$ is the static yield stress, $\gamma(\vartheta)$ is the viscosity coefficient of the material and ϑ stands for the temperature. The nonlinear low $\Phi(F)=F^{\delta}$ will be assumed where F denotes the static yield function and δ is a material constant. The elastic strains are omitted and the material is incompressible, $\dot{\varepsilon}_{ii}=\dot{\sigma}_{ii}/3K+\alpha\dot{\vartheta}=0$, where K is the modulus of volume expansion and α denotes the coefficient of thermal expansions. Experimental results confirmed a suitability description of the metals behaviour with only two quantities k and γ dependent on the temperature and Φ itself independent of it, see [14]. The physical equations of the Saint Venant–Levy–Mises theory of plastic flow, $\dot{\epsilon}_{ij}=\lambda s_{ij}$ are obtained from (2.1) if $\gamma=\infty$ and $J_2^{1/2}=k$. The functions $k(\vartheta)$ and $\gamma(\vartheta)$ for mild steel, established on the basis of experimental results by Maiden and Campbell[15] and Manjoine[16] have the form ($\delta = 5$)

$$k(\bar{\theta}) = 119.51 \exp\left[0.45\left(\frac{288}{\bar{\theta}} - 1\right)\right], \quad [\text{MPa}],$$

$$\gamma(\bar{\theta}) = 60.24\left[1 + 2.6\left(\frac{220 - \bar{\theta}}{273}\right)^2\right], \quad [\text{s}^{-1}] \tag{2.2}$$

where the temperature $\bar{\theta}$ is expressed in [K].

The shell operates at constant temperature and no constrains are imposed on the deformation. This results in no arising the stresses due to the coefficient of thermal expansion of the material. The applied impulse is regarded to be an accidental one.

The kinematic equations. According to the Kirchhoff–Love hypothesis the components of the strain rate tensor are, [17], Fig. 1:

$$\dot{\epsilon}_\phi = \frac{1}{a}\left(\frac{\partial \dot{u}}{\partial \phi} - \dot{w}\right) - \frac{z}{a^2}\left(\frac{\partial^2 \dot{w}}{\partial \phi^2} + \frac{\partial \dot{u}}{\partial \phi}\right),$$

$$\dot{\epsilon}_\theta = \frac{1}{a}\left(\frac{1}{\sin\phi}\frac{\partial \dot{v}}{\partial \phi} + \dot{u}\,\text{ctg}\,\phi - \dot{w}\right) - \frac{z}{a^2}\left(\frac{1}{\sin^2\phi}\frac{\partial^2 \dot{w}}{\partial \theta^2} + \frac{1}{\sin\phi}\frac{\partial \dot{v}}{\partial \theta} + \text{ctg}\,\phi\frac{\partial \dot{w}}{\partial \phi} + \dot{u}\,\text{ctg}\,\phi\right), \tag{2.3}$$

$$\dot{\epsilon}_{\phi\theta} = \dot{\epsilon}_{\theta\phi} = \frac{1}{2a}\left(\frac{\partial \dot{v}}{\partial \phi} + \frac{1}{\sin\phi}\frac{\partial \dot{u}}{\partial \theta} - \dot{v}\,\text{ctg}\,\phi\right) - \frac{z}{a^2}\left(\frac{1}{\sin\phi}\frac{\partial^2 \dot{w}}{\partial\phi\partial\theta} - \frac{\cos\phi}{\sin^2\phi}\frac{\partial \dot{w}}{\partial\theta}\right.$$

$$\left. + \frac{\sin\phi}{2}\frac{\partial}{\partial\phi}\left(\frac{\dot{v}}{\sin\phi}\right) + \frac{1}{2\sin\phi}\frac{\partial \dot{u}}{\partial\theta}\right).$$

The dynamic equilibrium equations. The following equations are employed[18], Fig. 1:

$$a\cos\phi N_\phi + a\sin\phi\frac{\partial N_\phi}{\partial\phi} + a\frac{\partial N_{\theta\phi}}{\partial\theta} - \tilde{r}_2 N_\theta - \tilde{r}_1 N_{\phi\theta} + \tilde{q}_1 Q_\phi + \tilde{p}_2 Q_\theta + a^2\sin\phi P_\phi = 0,$$

$$a\cos\phi N_{\theta\phi} + a\sin\phi\frac{\partial N_{\theta\phi}}{\partial\phi} + a\frac{\partial N_\theta}{\partial\theta} + \tilde{r}_1 N_\phi + \tilde{r}_2 N_{\theta\phi} - \tilde{q}_2 Q_\theta - \tilde{p}_1 Q_\phi + a^2\sin\phi P_\theta = 0,$$

$$a\cos\phi Q_\phi + a\sin\phi\frac{\partial Q_\phi}{\partial\phi} + a\frac{\partial Q_\theta}{\partial\theta} - \tilde{q}_1 N_\phi - \tilde{p}_2 N_{\theta\phi} + \tilde{p}_1 N_{\phi\theta} + \tilde{q}_2 N_\theta + a^2\sin\phi P_z = 0,$$

$$a\cos\phi M_{\phi\theta} + a\sin\phi\frac{\partial M_{\phi\theta}}{\partial\phi} - a\frac{\partial M_\theta}{\partial\theta} - \tilde{r}_1 M_\phi - \tilde{r}_2 M_{\theta\phi} + a^2\sin\phi Q_\theta = 0,$$

$$a\cos\phi M_\phi + a\sin\phi\frac{\partial M_\phi}{\partial\phi} + a\frac{\partial M_{\theta\phi}}{\partial\theta} + \tilde{r}_1 M_{\phi\theta} - \tilde{r}_2 M_\theta - a^2\sin\phi Q_\phi = 0,$$

$$\tilde{p}_1 M_\phi + \tilde{p}_2 M_\theta - \tilde{q}_1 M_{\phi\theta} + \tilde{q}_2 M_{\theta\phi} + a^2\sin\phi(N_{\phi\theta} - N_{\theta\phi}) = 0 \tag{2.4}$$

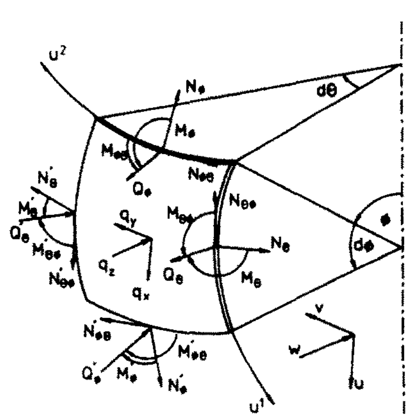

Fig. 1. Shell element, sign convention.

where

$$\tilde{p}_1 = \frac{\partial^2 w}{\partial\phi\partial\theta} - \text{ctg}\,\phi\frac{\partial w}{\partial\theta},$$

$$\tilde{p}_2 = -\frac{\partial}{\partial\theta}\left(\frac{\partial w}{\partial\phi} + u\right) + \text{ctg}\,\phi\left(\frac{\partial w}{\partial\theta} + v\sin\phi\right) - \sin\phi\frac{\partial v}{\partial\phi},$$

$$\tilde{q}_1 = -\sin\phi\left[a + \frac{\partial}{\partial\phi}\left(\frac{\partial w}{\partial\phi} + u\right)\right],$$

$$\tilde{q}_2 = a\sin\phi + \frac{\partial}{\partial\theta}\left(\frac{1}{\sin\phi}\frac{\partial w}{\partial\theta} + v\right) + \cos\phi\left(\frac{\partial w}{\partial\phi} + u\right), \tag{2.5}$$

$$\tilde{r}_1 = \sin\phi\left(\frac{\partial^2 v}{\partial\phi^2} + v\right) + \frac{\partial w}{\partial\theta},$$

$$\tilde{r}_2 = a\cos\phi + \frac{\partial^2 v}{\partial\phi\partial\theta} - \sin\phi\left(\frac{\partial w}{\partial\phi} + u\right),$$

$$P_\phi = q_\phi - \rho h\ddot{u}, \quad P_\theta = q_\theta - \rho h\ddot{v}, \quad P_z = q_z - \rho h\ddot{w}, \tag{2.6}$$

ρ is the density of the material and h denotes the thickness.

The internal resultant forces and moments are defined as follows:

$$N_{\substack{\phi\\\theta}} = \int_{-h/2}^{h/2}\sigma_{\substack{\phi\\\theta}}\,\text{d}z, \qquad N_{\phi\theta} = N_{\theta\phi} = \int_{-h/2}^{h/2}\sigma_{\phi\theta}\,\text{d}z,$$

$$M_{\substack{\phi\\\theta}} = \int_{-h/2}^{h/2}\sigma_{\substack{\phi\\\theta}}z\,\text{d}z, \qquad M_{\phi\theta} = -M_{\theta\phi} = -\int_{-h/2}^{h/2}\sigma_{\phi\theta}z\,\text{d}z. \tag{2.7}$$

3. METHOD OF SOLUTION

Assuming $\sigma_{33} = 0$, $\dot{\epsilon}_{13} = \dot{\epsilon}_{23} = 0$, according to the Kirchhoff–Love theory of shells, we get from (2.1), for $J_2^{1/2} > k$, the following nonlinear equations:

$$\sigma_{\alpha\beta} = A(\dot{\epsilon}_{\alpha\beta} + \dot{\epsilon}_{\rho\rho}\delta_{\alpha\beta}), \qquad \alpha, \beta, \rho = 1, 2 \tag{3.1}$$

where

$$A = \frac{k}{\dot{I}_2^{1/2}}\left[1 + \left(\frac{2}{\gamma}\dot{I}_2^{1/2}\right)^{1/\delta}\right], \qquad \dot{I}_2 = \frac{1}{2}(\dot{\epsilon}_{\alpha\beta} + \dot{\epsilon}_{\rho\rho}\delta_{\alpha\beta})\dot{\epsilon}_{\alpha\beta}. \tag{3.2a,b}$$

$\dot{I}_2$ is the second invariant of the strain rate deviator and $\delta_{\alpha\beta}$ denotes the Kronecker delta. The loading criterion $J_2^{1/2} > k$ is equivalent to $\dot{I}_2 > 0$.

In order to obtain the internal resultant forces and moments (2.7), the yield rules should be transformed into the space of forces and moments. Using eqns (2.7) and (3.1), we obtain the integrals of irrational expressions which cannot be shown in the explicit form. Thus, other way of solution will be sought.

A characteristic feature of dynamic buckling is the significance of inertial effects in restraining the growth of buckling mode amplitudes at an early stage of the motion. These effects result in the yielding before the instabilities become dominant. Analytically, the problem can be formulated as a superposition of small perturbations $u_p(\phi, \theta, t)$, $v_p(\phi, \theta, t)$, $w_p(\phi, \theta, t)$ on the basic unperturbed motion $u_0 = v_0 = 0$, $w_0(t)$. The amplitudes of perturbed displacements are restricted to be so small that the homogeneous compressive deformation predominates over local bending. Also, this condition permits the constitutive equations to be linearized by the expansion of eqns (3.1) into Taylor's series in three variables in the vicinity of unperturbed motion and by retaining two terms only. We obtain:

$$\sigma_{\alpha\beta} = A^0(\dot\epsilon_{\alpha\beta} + \dot\epsilon_{\rho\rho}\delta_{\alpha\beta}) + B^0[(\dot\epsilon^0_{\eta\omega} + \dot\epsilon^0_{\rho\rho}\delta_{\eta\omega})(\dot\epsilon_{\eta\omega} - \dot\epsilon^0_{\eta\omega})](\dot\epsilon^0_{\alpha\beta} + \dot\epsilon^0_{\mu\mu}\delta_{\alpha\beta}) \tag{3.3a}$$

where

$$A^0 = \frac{k}{\sqrt{\dot I_2^0}}\left[1 + \left(\frac{2}{\gamma}\sqrt{\dot I_2^0}\right)^{1/\delta}\right], \quad B^0 = \frac{k}{2}(\dot I_2^0)^{-3/2}\left[-1 + \frac{1-\delta}{\delta}\left(\frac{2}{\gamma}\sqrt{\dot I_2^0}\right)^{1/\delta}\right], \tag{3.3b}$$

$$\dot I_2^0 = \frac{1}{2}(\dot\epsilon^0_{\beta\eta} + \dot\epsilon^0_{\alpha\alpha}\delta_{\beta\eta})\dot\epsilon^0_{\beta\eta}, \quad \alpha, \beta, \ldots = 1, 2. \tag{3.3c}$$

The eqns (3.3a) can be rewritten in the form

$$\sigma_{\alpha\beta} = \sigma^0_{\alpha\beta} + \sigma^p_{\alpha\beta} \tag{3.3d}$$

where

$$\sigma^0_{\alpha\beta} = A^0(\dot\epsilon^0_{\alpha\beta} + \dot\epsilon^0_{\rho\rho}\delta_{\alpha\beta}), \tag{3.3e}$$

$$\sigma^p_{\alpha\beta} = A^0(\dot\epsilon^p_{\alpha\beta} + \dot\epsilon^p_{\rho\rho}\delta_{\alpha\beta}) + B^0[(\dot\epsilon^0_{\eta\omega} + \dot\epsilon^0_{\rho\rho}\delta_{\eta\omega})\dot\epsilon^p_{\eta\omega}](\dot\epsilon^0_{\alpha\beta} + \dot\epsilon^0_{\mu\mu}\delta_{\alpha\beta}), \tag{3.3f}$$

$$\dot\epsilon_{\alpha\beta} = \dot\epsilon^0_{\alpha\beta} + \dot\epsilon^p_{\alpha\beta}. \tag{3.3g}$$

The indices 0 and p denote the unperturbed and perturbed quantities, respectively. Equations (3.3) may now be used together with eqns (2.3), (2.4) and (2.7) to obtain the differential equations governing the viscoplastic flow buckling of the shell.

Buckling stems from the growth of small imperfections in the otherwise uniform displacement and loading fields. It turns out that certain harmonics grow rapidly and cause the shell to exhibit a characteristic wrinkled shape which is characterized by critical mode numbers. This property of the amplitudes is used to determine the threshold impulse that the shell can tolerate without excessive deformation.

4. UNPERTURBED MOTION

The displacements are expressed by the functions

$$u_0 = 0, \quad v_0 = 0, \quad w_0 = w_0(t). \tag{4.1}$$

Substituting (4.1) into (2.3) leads to the components

$$\dot\epsilon_\phi{}^0 = \dot\epsilon_\theta{}^0 = -\frac{\dot w_0}{a}, \quad \dot\epsilon_{\phi\theta} = \dot\epsilon_{\theta\phi} = 0 \tag{4.2}$$

and hence by eqns (3.3e) and (2.7) the following formulae are obtained

$$\sigma_\phi{}^0 = \sigma_\theta{}^0 = -3\frac{C_0}{h}\dot w_0, \qquad \sigma^0_{\phi\theta} = \sigma^0_{\theta\phi} = 0, \tag{4.3}$$

$$N_\phi{}^0 = N_\theta{}^0 = -3C_0\dot w_0, \qquad N^0_{\phi\theta} = N^0_{\theta\phi} = 0, \qquad M^0_{\phi\theta} = M^0_{\theta\phi} = 0 \tag{4.4}$$

where

$$C_0 = \frac{h}{a}A^0 = \frac{kh}{\sqrt{3}\dot w_0}\left[1 + \left(\frac{2\sqrt{3}\dot w_0}{\gamma a}\right)^{1/\delta}\right]. \tag{4.5}$$

In such a situation eqns (2.4) reduce to the equation

$$N_\phi^0 + N_\theta^0 + aP_z^0 = 0, \qquad P_z^0 = q_z^0(t) - \rho h \ddot{w}_0 \tag{4.6}$$

which, combined with the expressions (4.4), yields

$$\ddot{w}_0 + c\dot{w}_0^{1/\delta} = \alpha \tag{4.7}$$

where

$$c = \frac{k}{\rho}\left(\frac{1}{\gamma}\right)^{1/\delta}\left(\frac{2\sqrt{3}}{a}\right)^{(1+\delta)/\delta}, \qquad \alpha = \frac{1}{\rho h}\left(q_z^0 - \frac{2\sqrt{3}kh}{a}\right). \tag{4.8}$$

This nonlinear equation for the given initial conditions has a closed form solution, but for $\delta \neq 1$ it is impossible to find an explicit form. Here, the parametric form will be presented for two types of the impulse.

In the case of rectangular impulse pressure $q_z^0 = Q$ for $0 \leq t \leq T$ and $q_z^0 = 0$ for $t > T$, T is the duration time of the load and for the initial conditions $w_0(0) = 0$, $\dot{w}_0(0) = 0$ the solution of eqn (4.7) can be given in the following parametric form

$$w_0^*(\xi) = w_0(t(\xi)), \qquad \frac{\mathrm{d}w_0^*(\xi)}{\mathrm{d}t} = \xi^\delta \tag{4.9}$$

where $t(\xi)$ is obtained from eqn (4.7)

$$t(\xi) = \frac{\delta}{\alpha}\int_0^\xi \frac{\xi^{\delta-1}}{1-\xi c/\alpha}\,\mathrm{d}\xi = -\frac{\delta\alpha^{\delta-1}}{c^\delta}\left[\sum_{i=1}^{\delta-1}\frac{1}{\delta-i}\left(\frac{c}{\alpha}\xi\right)^{\delta-i} + \ln\left(1-\frac{c}{\alpha}\xi\right)\right],$$
$$0 \leq \xi \leq \alpha/c, \quad t \geq 0, \tag{4.10}$$

and $w_0^*(\xi)$ from (4.9) and (4.10),

$$w_0^*(\xi) = \frac{\delta}{\alpha}\int_0^\xi \frac{\xi^{2\delta-1}}{1-\xi c/\alpha}\,\mathrm{d}\xi = -\frac{\delta\alpha^{2\delta-1}}{c^{2\delta}}\left[\sum_{i=1}^{2\delta-1}\frac{1}{2\delta-i}\left(\frac{c}{\alpha}\xi\right)^{2\delta-i} + \ln\left(1-\frac{c}{\alpha}\xi\right)\right]. \tag{4.11}$$

The solution (4.11) is valid for $0 \leq t \leq T$ or $0 \leq \xi \leq \xi_T$ where $t(\xi_T) = T$.

For $t \geq T$ it appears convenient to express the eqn (4.7) in the form

$$\ddot{w}_0 + c\dot{w}^{1/\delta} = -\beta, \qquad \beta = \frac{2\sqrt{3}k}{a\rho} \tag{4.12}$$

and to introduce a new parameter ζ,

$$w_0^*(\zeta) = w_0(t(\zeta)), \qquad \frac{\mathrm{d}w_0^*(\zeta)}{\mathrm{d}t} = (-\zeta)^\delta. \tag{4.13}$$

From the continuity condition it results that $\zeta_T = -\xi_T$ and the solution is of the form

$$t(\zeta) = (-1)^{\delta-1}\frac{\delta}{\beta}\int_{\zeta_T}^{\zeta}\frac{\zeta^{\delta-1}}{1-\zeta c/\beta}\,\mathrm{d}\zeta + T$$
$$= (-1)^\delta\frac{\delta\beta^{\delta-1}}{c^\delta}\left\{\sum_{i=1}^{\delta-1}\frac{1}{\delta-i}\left[\left(\frac{c}{\beta}\zeta\right)^{\delta-i} - \left(-\frac{c}{\beta}\xi_T\right)^{\delta-i}\right] + \ln\frac{1-\zeta c/\beta}{1+\xi_T c/\beta}\right\} + T, \tag{4.14}$$
$$\zeta_T \leq \zeta \leq \beta/c,$$

$$w_0^*(\zeta) = (-1)^\delta \frac{\delta}{\beta}\int_{\zeta_T}^{\zeta} \frac{\zeta^{2\delta-1}}{1-\zeta c/\beta}\,\mathrm{d}\zeta + w_0(T)$$

$$= (-1)^{\delta-1}\frac{\delta\beta^{2\delta-1}}{c^{2\delta}}\left\{\sum_{i=1}^{2\delta-1}\frac{1}{2\delta-i}\left[\left(\frac{c}{\beta}\zeta\right)^{2\delta-i} - \left(-\frac{c}{\beta}\xi_T\right)^{2\delta-i}\right] + \ln\frac{1-\zeta c/\beta}{1+\xi_T c/\beta}\right\} + w_0(T) \tag{4.15}$$

where in (4.14), (4.15) $T = t(\xi_T)$, $w_0(T) = w_0^*(\xi_T)$ are given in the parametric form by the solutions (4.10), (4.11). Refining the variation range of ζ, we have $-\xi_T \leqslant \zeta \leqslant 0$. The unperturbed motion ceases at the instant $t = t_f$, $(\zeta_f = 0)$, when $\dot{w}_0(t) = 0$. Thus

$$t_f = (-1)^{\delta+1}\frac{\delta\beta^{\delta-1}}{c^\delta}\left[\sum_{i=1}^{\delta-1}\frac{1}{\delta-i}\left(-\frac{c}{\beta}\xi_T\right)^{\delta-i} + \ln\left(1+\frac{c}{\beta}\xi_T\right)\right] + T \tag{4.16}$$

and the final displacements are

$$w_{0f} = (-1)^\delta\frac{\delta\beta^{2\delta-1}}{c^{2\delta}}\left[\sum_{i=1}^{2\delta-1}\frac{1}{2\delta-i}\left(-\frac{c}{\beta}\xi_T\right)^{2\delta-i} + \ln\left(1+\frac{c}{\beta}\xi_T\right)\right] + w_0(T). \tag{4.17}$$

For the linear case ($\delta = 1$), the solution of eqn (4.7) can be expressed in the explicit form[12]. From eqns (4.10), (4.11) we obtain for $0 \leqslant t \leqslant T$

$$\xi(t) = \frac{\alpha}{c}(1-\mathrm{e}^{-ct}),$$

$$w_0(t) = -\frac{\alpha}{c}\left[\frac{1}{c}(1-\mathrm{e}^{-ct}) - t\right]. \tag{4.18}$$

From eqns (4.14)–(4.16) we get for $T \leqslant t \leqslant t_f$

$$\zeta(t) = \frac{\beta}{c}\left\{1 - \left[1+\frac{\alpha}{\beta}(1-\mathrm{e}^{-cT})\right]\mathrm{e}^{-c(t-T)}\right\},$$

$$w_0(t) = \frac{\beta}{c^2}(1-ct-\mathrm{e}^{-ct}) + \frac{\alpha+\beta}{c}\left[T + \frac{1}{c}(1-\mathrm{e}^{-cT})\mathrm{e}^{-ct}\right], \tag{4.19}$$

$$t_f = \frac{1}{c}\ln\left[1 - \frac{\alpha+\beta}{\beta}(1-\mathrm{e}^{-cT})\right].$$

In the case of an ideal impulse pressure (uniform initial velocity V_0) the initial conditions are

$$w_0(0) = 0, \qquad \dot{w}_0(0) = V_0$$

and the solution of eqn (4.12) has the form

$$t(\zeta) = (-1)^\delta\frac{\delta\beta^{\delta-1}}{c^\delta}\left\{\sum_{i=1}^{\delta-1}\frac{1}{\delta-i}\left[\left(\frac{c}{\beta}\zeta\right)^{\delta-i} - \left(-\frac{c}{\beta}V_0^{1/\delta}\right)^{\delta-i}\right] + \ln\frac{1-\zeta c/\beta}{1+V_0^{1/\delta}c/\beta}\right\},$$

$$w_0^*(\zeta) = (-1)^{\delta-1}\frac{\delta\beta^{2\delta-1}}{c^{2\delta}}\left\{\sum_{i=1}^{2\delta-1}\frac{1}{2\delta-i}\left[\left(\frac{c}{\beta}\zeta\right)^{2\delta-i} - \left(-\frac{c}{\beta}V_0^{1/\delta}\right)^{2\delta-i}\right]\right.$$

$$\left. + \ln\frac{1-\zeta c/\beta}{1+V_0^{1/\delta}c/\beta}\right\} \tag{4.20}$$

where $w_0^*(\zeta) = w_0(t(\zeta))$ and $-V_0^{1/\delta} \leqslant \zeta \leqslant 0$. For $\zeta = 0$, the final values of t_f and w_{0f}^* can be easily obtained.

For $\delta = 1$ we get

$$w_0(t) = \frac{\beta}{c^2}\left[1 + \frac{c}{\beta}V_0 - ct - \left(1 + \frac{c}{\beta}V_0\right)e^{-ct}\right],$$

$$t_f = \frac{\beta}{c^2}\left[\frac{c}{\beta}V_0 - \ln\left(1 + \frac{c}{\beta}V_0\right)\right]. \tag{4.21}$$

5. PERTURBED MOTION

The total displacements in the perturbed motion are expressed by the following functions

$$u = u_p(\phi, \theta, t), \quad v = v_p(\phi, \theta, t), \quad w = w_0(t) + w_p(\phi, \theta, t). \tag{5.1}$$

Accounting for (5.1) in eqns (2.3) leads to the expressions:

$$\begin{aligned}
\dot{\epsilon}_\phi &= \dot{\epsilon}_\phi{}^0 + \frac{1}{a}\left(\frac{\partial \dot{u}_p}{\partial \phi} - \dot{w}_p\right) - \frac{z}{a^2}\left(\frac{\partial^2 \dot{w}_p}{\partial \phi^2} + \frac{\partial \dot{u}_p}{\partial \phi}\right),\\
\dot{\epsilon}_\theta &= \dot{\epsilon}_\theta{}^0 + \frac{1}{a}\left(\frac{1}{\sin\phi}\frac{\partial \dot{v}_p}{\partial \theta} + \dot{u}_p \operatorname{ctg}\phi - \dot{w}_p\right) - \frac{z}{a^2}\left(\frac{1}{\sin^2\phi}\frac{\partial^2 \dot{w}_p}{\partial \theta^2} + \frac{1}{\sin\phi}\frac{\partial \dot{v}_p}{\partial \theta}\right.\\
&\quad \left. + \operatorname{ctg}\phi \frac{\partial \dot{w}_p}{\partial \phi} + \dot{u}_p \operatorname{ctg}\phi\right),\\
\dot{\epsilon}_{\phi\theta} &= \dot{\epsilon}_{\theta\phi} = \frac{1}{2a}\left(\frac{\partial \dot{v}_p}{\partial \phi} + \frac{1}{\sin\phi}\frac{\partial \dot{u}_p}{\partial \theta} - \dot{v}_p \operatorname{ctg}\phi\right) - \frac{z}{a^2}\left(\frac{1}{\sin\phi}\frac{\partial^2 \dot{w}_p}{\partial\phi\partial\theta} - \frac{\cos\phi}{\sin^2\phi}\frac{\partial \dot{w}_p}{\partial \theta}\right.\\
&\quad \left. + \frac{1}{2}\sin\phi\frac{\partial}{\partial\phi}\left(\frac{\dot{v}_p}{\sin\phi}\right) + \frac{1}{2\sin\phi}\frac{\partial \dot{u}_p}{\partial \theta}\right).
\end{aligned} \tag{5.2}$$

From eqns (3.3) and (5.2) the stress components are obtained,

$$\begin{aligned}
\sigma_\alpha &= \sigma_\alpha{}^0 + \frac{C_\alpha}{h}\left[\frac{\partial \dot{u}_p}{\partial \phi} - \frac{z}{a}\frac{\partial}{\partial \phi}\left(\frac{\partial \dot{w}_p}{\partial \phi} + \dot{u}_p\right)\right] + \frac{C_{\bar{\gamma}}}{h}\left\{\frac{1}{\sin\phi}\left(\frac{\partial \dot{v}_p}{\partial \theta} + \dot{u}_p\cos\phi\right)\right.\\
&\quad \left. - \frac{z}{a}\left[\frac{1}{\sin^2\phi}\frac{\partial}{\partial\theta}\left(\frac{\partial \dot{w}_p}{\partial \theta} + \dot{v}_p\sin\phi\right) + \left(\frac{\partial \dot{w}_p}{\partial\phi} + \dot{u}_p\right)\operatorname{ctg}\phi\right]\right\} + \frac{C_3}{h}\dot{w}_p,\\
\sigma_{\phi\theta} &= \sigma_{\theta\phi} = \frac{C_0}{2h}\left[\frac{1}{\sin\phi}\left(\sin\phi\frac{\partial \dot{v}_p}{\partial \phi} + \frac{\partial \dot{u}_p}{\partial \theta} - \dot{v}_p\cos\phi\right) - \frac{z}{a\sin\phi}\right.\\
&\quad \left. \times\left(-2\operatorname{ctg}\phi\frac{\partial \dot{w}_p}{\partial \theta} + 2\frac{\partial^2 \dot{w}_p}{\partial\phi\partial\theta} + \frac{\partial \dot{u}_p}{\partial \theta} - \dot{v}_p\cos\phi + \sin\phi\frac{\partial \dot{v}_p}{\partial \phi}\right)\right]
\end{aligned} \tag{5.3}$$

where $\alpha = 1, 2$ or ϕ, θ, $\bar{\gamma} = 3 - \alpha$ and C_0 is given by (4.5),

$$\begin{aligned}
C_1 &= \frac{kh}{2\sqrt{3}\dot{w}_0}\left[1 + \frac{3+\delta}{\delta}\left(\frac{2\sqrt{3}\dot{w}_0}{\gamma a}\right)^{1/\delta}\right],\\
C_2 &= \frac{kh}{2\sqrt{3}\dot{w}_0}\left[-1 + \frac{3-\delta}{\delta}\left(\frac{2\sqrt{3}\dot{w}_0}{\gamma a}\right)^{1/\delta}\right],\\
C_3 &= -\frac{3kh}{\delta a}\left(\frac{2}{\gamma}\right)^{1/\delta}\left(\frac{\sqrt{3}\dot{w}_0}{a}\right)^{(1-\delta)/\delta}.
\end{aligned} \tag{5.4}$$

The stress components (5.3) produce, according to (2.7), the following resultant forces and moments:

$$N_\alpha = N_\alpha^0 + C_3\dot{w}_p + C_\alpha\frac{\partial\dot{u}_p}{\partial\phi} + \frac{C_V}{\sin\phi}\left(\frac{\partial\dot{v}_p}{\partial\theta} + \dot{u}_p\cos\phi\right),$$

$$N_{\phi\theta} = N_{\theta\phi} = \frac{C_0}{2\sin\phi}\left(\sin\phi\frac{\partial\dot{v}_p}{\partial\phi} + \frac{\partial\dot{u}_p}{\partial\theta} - \dot{v}_p\cos\phi\right), \tag{5.5}$$

$$M_\alpha = -\frac{h^2}{12a}\left\{C_\alpha\frac{\partial}{\partial\phi}\left(\frac{\partial\dot{w}_p}{\partial\phi} + \dot{u}_p\right) + C_V\left[\frac{1}{\sin^2\phi}\frac{\partial}{\partial\theta}\left(\frac{\partial\dot{w}_p}{\partial\theta} + \dot{v}_p\sin\phi\right)\right.\right.$$
$$\left.\left.+ \operatorname{ctg}\phi\left(\frac{\partial\dot{w}_p}{\partial\phi} + \dot{u}_p\right)\right]\right\},$$

$$M_{\phi\theta} = -M_{\theta\phi} = \frac{h^2}{24a}\frac{C_0}{\sin\phi}\left(-2\operatorname{ctg}\phi\frac{\partial\dot{w}_p}{\partial\theta} + 2\frac{\partial^2\dot{w}_p}{\partial\phi\partial\theta} + \frac{\partial\dot{u}_p}{\partial\theta} - \dot{v}_p\cos\phi + \sin\phi\frac{\partial\dot{v}_p}{\partial\phi}\right).$$

Eliminating shear forces from eqns (2.4), neglecting the terms with products of perturbation quantities and accounting for the sixth equation being satisfied identically, the dynamic equilibrium equations can be reduced to the form

$$a\cos\phi(N_\phi - N_\theta) + a\sin\phi\frac{\partial N_\phi}{\partial\phi} + a\frac{\partial N_{\theta\phi}}{\partial\theta} - \cos\phi(M_\phi - M_\theta) - \sin\phi\frac{\partial M_\phi}{\partial\phi} - \frac{\partial M_{\theta\phi}}{\partial\theta}$$
$$-\left[\frac{\partial^2 v}{\partial\phi\partial\theta} - \sin\phi\left(\frac{\partial w}{\partial\phi} + u\right)\right]N_\theta^0 + a^2\sin\phi P_\phi = 0,$$

$$2a\cos\phi N_{\theta\phi} + a\sin\phi\frac{\partial N_{\theta\phi}}{\partial\phi} + a\frac{\partial N_\theta}{\partial\theta} + \cos\phi(M_{\phi\theta} - M_{\theta\phi}) + \sin\phi\frac{\partial M_{\phi\theta}}{\partial\phi} - \frac{\partial M_\theta}{\partial\theta}$$
$$+\left[\sin\phi\left(\frac{\partial^2 v}{\partial\phi^2} + v\right) + \frac{\partial w}{\partial\theta}\right]N_\phi^0 + a^2\sin\phi P_\theta = 0, \tag{5.6}$$

$$\sin\phi\left[a(N_\phi + N_\theta) - M_\phi + M_\theta + \frac{\partial^2 M_\phi}{\partial\phi^2}\right] + \cos\phi\frac{\partial}{\partial\phi}(2M_\phi - M_\theta) + \left(\operatorname{ctg}\phi\frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial\phi\partial\theta}\right)$$
$$\times(M_{\theta\phi} - M_{\phi\theta}) + \frac{1}{\sin\phi}\frac{\partial^2 M_\theta}{\partial\theta^2} + \sin\phi\frac{\partial}{\partial\phi}\left(\frac{\partial w}{\partial\phi} + u\right)N_\phi^0$$
$$+\left[\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\phi}\frac{\partial w}{\partial\theta} + v\right) + \cos\phi\left(\frac{\partial w}{\partial\phi} + u\right)\right]N_\theta^0 + a^2\sin\phi P_z = 0.$$

On substituting (5.5) into (5.6) and accounting for (4.6), the three governing equations of the viscoplastic flow buckling are obtained. These partial differential equations of the fifth order with variable coefficients can be expressed in the form

$$T_{ij}(u_p^j) + \tilde{T}_{ij}(\dot{u}_p^j) + aP_i^p = 0 \tag{5.7}$$

where $j = 1, 2, 3$ is the summation index, $u_p^1 = u_p$, $u_p^2 = v_p$, $u_p^3 = w_p$, P_i^p is given by (2.6), $i = 1, 2, 3$ or ϕ, θ, z and $\tilde{T}_{ij}$ denote the following differential operators:

$$T_{11} = -3C_0\frac{\dot{w}_0}{a},\quad T_{12} = 0,\quad T_{13} = -3C_0\frac{\dot{w}_0}{a}\frac{\partial}{\partial\phi},$$

$$T_{21}=0,\quad T_{22}=-3C_0\frac{\dot{w}_0}{a},\quad T_{23}=-3C_0\frac{\dot{w}_0}{a}\frac{1}{\sin\phi}\frac{\partial}{\partial\theta},$$

$$T_{31}=-3C_0\frac{\dot{w}_0}{a}\left(\operatorname{ctg}\phi+\frac{\partial}{\partial\phi}\right),\quad T_{32}=-3C_0\frac{\dot{w}_0}{a}\frac{1}{\sin\phi}\frac{\partial}{\partial\theta},\quad T_{33}=-3C_0\frac{\dot{w}_0}{a}\nabla^2,$$

$$\tilde{T}_{11}=C_1(1-\operatorname{ctg}^2\phi+\nabla^2)+\frac{C_3}{2}\left(2+\frac{1}{\sin^2\phi}\frac{\partial^2}{\partial\theta^2}\right),$$

$$\tilde{T}_{12}=-C_0\frac{\operatorname{ctg}\phi}{\sin\phi}\frac{\partial}{\partial\theta}+\frac{C_3}{2\sin\phi}\left(\operatorname{ctg}\phi-\frac{\partial}{\partial\phi}\right)\frac{\partial}{\partial\theta},$$

$$\tilde{T}_{13}=\bar{c}C_1\left(\frac{\partial\nabla^2}{\partial\phi}+2\frac{\partial}{\partial\phi}\right)+C_3\frac{\partial}{\partial\phi},\tag{5.8}$$

$$\tilde{T}_{21}=C_0\frac{\operatorname{ctg}\phi}{\sin\phi}\frac{\partial}{\partial\theta}-\frac{C_3}{2\sin\phi}\left(\operatorname{ctg}\phi+\frac{\partial}{\partial\phi}\right)\frac{\partial}{\partial\theta},$$

$$\tilde{T}_{22}=\frac{C_0}{2}(1-\operatorname{ctg}^2\phi+\nabla^2)-\frac{C_3}{2\sin\phi}\frac{\partial^2}{\partial\theta^2},$$

$$\tilde{T}_{23}=\frac{\bar{c}C_1}{\sin\phi}\left(\frac{\partial\nabla^2}{\partial\theta}+2\frac{\partial}{\partial\theta}\right)+\frac{C_3}{\sin\phi}\frac{\partial}{\partial\theta},$$

$$\tilde{T}_{31}=-\bar{c}C_1\left[\frac{\partial\nabla^2}{\partial\phi}+\operatorname{ctg}\phi\frac{\partial^2}{\partial\phi^2}+\frac{\partial}{\partial\phi}+(3+\operatorname{ctg}^2\phi)\operatorname{ctg}\phi+\frac{3\operatorname{ctg}\phi}{\sin^2\phi}\frac{\partial^2}{\partial\theta^2}\right]-C_3\left(\operatorname{ctg}\phi+\frac{\partial}{\partial\phi}\right),$$

$$\tilde{T}_{32}=\frac{\bar{c}C_1}{\sin\phi}\left(-\frac{\partial\nabla^2}{\partial\theta}+3\operatorname{ctg}\phi\frac{\partial^2}{\partial\phi\partial\theta}-\frac{3+\operatorname{ctg}^2\phi}{\sin\phi}\frac{\partial}{\partial\theta}\right)-\frac{C_3}{\sin\phi}\frac{\partial}{\partial\theta},$$

$$\tilde{T}_{33}=-\bar{c}C_1\nabla^2(\nabla^2+1)+C_3(2-\bar{c}\nabla^2)$$

and

$$\nabla^2=\frac{\partial^2}{\partial\phi^2}+\operatorname{ctg}\phi\frac{\partial}{\partial\phi}+\frac{1}{\sin^2\phi}\frac{\partial^2}{\partial\theta^2},\quad \bar{c}=\frac{h^2}{12a^2}.$$

The terms $h^2/12a^2$ occurring in the first and the second equation of (5.7) have been disregarded as much smaller than unity.

In order to obtain the solution in terms of the spherical functions, the eqns (5.7) will be transformed into the corresponding form. Differentiating the first of eqns (5.7), ($i=1$) with respect to θ, multiplying the second ($i=2$) by $-\sin\phi$ and next differentiating it with respect to ϕ, and adding them together we eventually find, introducing a new variable

$$\mu=\frac{1}{\sin\phi}\frac{\partial u_p}{\partial\theta}-\frac{\partial v_p}{\partial\phi}-v_p\operatorname{ctg}\phi\tag{5.9}$$

that

$$\frac{C_0}{2}\left(L\dot{\mu}-6\frac{\dot{w}_0}{a}\mu\right)+a(q_\mu-\rho h\ddot{\mu})=0\tag{5.10}$$

where

$$q_\mu = \frac{1}{\sin\phi}\frac{\partial q_\phi^p}{\partial\theta} - \frac{\partial q_\theta^p}{\partial\phi} - q_\theta^p \operatorname{ctg}\phi, \quad L = \nabla^2 + 2. \tag{5.11}$$

A similar operation will be made on the same equations in the next step. Multiplying the first equation by $\sin\phi$ and differentiating it with respect to ϕ, differentiating the second with respect to θ, adding them together and dividing the result by $\sin\phi$, we obtain

$$C_1 L[\dot\eta + \bar c(L-2)\dot w_p] + C_3[\dot\eta + (L-2)\dot w_p] - 3C_0\frac{\dot w_0}{a}[\eta + (L-2)w_p] + a(q_\eta - \rho h\ddot\eta) = 0 \tag{5.12}$$

where

$$\eta = \frac{\partial u_p}{\partial\phi} + u_p \operatorname{ctg}\phi + \frac{1}{\sin\phi}\frac{\partial v_p}{\partial\theta}, \tag{5.13}$$

$$q_\eta = \frac{\partial q_\phi^p}{\partial\phi} + q_\phi^p \operatorname{ctg}\phi + \frac{1}{\sin\phi}\frac{\partial q_\theta^p}{\partial\theta}. \tag{5.14}$$

The third equation (5.7), ($i = 3$) expressed in terms of unknown functions μ, η and w_p has the form

$$-\bar c C_1 L[\dot\eta + (L-2)\dot w_p] - C_3[\dot\eta + (\bar c L - 2)\dot w_p]$$
$$- 3C_0\frac{\dot w_0}{a}[\eta + (L-2)w_p] + a(q_z^p - \rho h\ddot w_p) = 0. \tag{5.15}$$

The set of equations (5.10), (5.12) and (5.15) can be reduced to the ordinary differential equations by taking the following series:

$$\begin{matrix} u_p \\ q_\phi^p \end{matrix} = \sum_{n=0}^{\infty}\sum_{m=0}^{n}\left[\begin{matrix} u_{mn}(t) \\ q_{mn}^\phi(t) \end{matrix}\frac{\mathrm{d}P_n^m}{\mathrm{d}\phi} + \begin{matrix} v_{mn}(t) \\ q_{mn}^\theta(t) \end{matrix}\frac{m}{\sin\phi}P_n^m\right]\cos m\theta, \tag{5.16a}$$

$$\begin{matrix} v_p \\ q_\theta^p \end{matrix} = \sum_{n=0}^{\infty}\sum_{m=0}^{n}\left[\begin{matrix} -u_{mn}(t) \\ q_{mn}^\phi(t) \end{matrix}\frac{m}{\sin\phi}P_n^m + \begin{matrix} -v_{mn}(t) \\ q_{mn}^\theta(t) \end{matrix}\frac{\mathrm{d}P_n^m}{\mathrm{d}\phi}\right]\sin m\theta, \tag{5.16b}$$

$$\begin{matrix} w_p \\ q_z^p \end{matrix} = \sum_{n=0}^{\infty}\sum_{m=0}^{n}\begin{matrix} w_{mn}(t) \\ q_{mn}^z(t) \end{matrix} P_n^m \cos m\theta \tag{5.16c}$$

where $P_n^m(\cos\phi)$ are the associated Legendre polynomials of degree n and order m. Substituting (5.16) into (5.9) ÷ (5.15) and accounting for $LP_n^m = -\lambda_n P_n^m$, $LLP_n^m = \lambda_n^2 P_n^m$ where $\lambda_n = n(n+1) - 2$, leads to the following equations for the amplitudes $u_{mn}(t)$, $v_{mn}(t)$ and $w_{mn}(t)$:

$$\ddot v_{mn} + \frac{1}{a\rho h}\left[\frac{C_0}{2}\left(\lambda_n \dot v_{mn} + 6\frac{\dot w_0}{a}v_{mn}\right) - aq_{mn}^\theta\right] = 0, \tag{5.17a}$$

$$\ddot u_{mn} + \frac{1}{a\rho h}\left[\lambda_n C_1(\dot u_{mn} + \bar c\dot w_{mn}) - C_3(\dot u_{mn} + \dot w_{mn}) + 3\frac{\dot w_0}{a}C_0(u_{mn} + w_{mn}) - aq_{mn}^\phi\right] = 0, \tag{5.17b}$$

$$\ddot w_{mn} - \frac{1}{a\rho h}\left[(\lambda_n + 2)(C_3 - \lambda_n \bar c C_1)\dot u_{mn} + \bar c(\lambda_n + 2)(C_3 - \lambda_n C_1)\dot w_{mn} + 2C_3\dot w_{mn} + 3(\lambda_n + 2)\right.$$
$$\left.\times\frac{\dot w_0}{a}C_0(u_{mn} + w_{mn}) + aq_{mn}^z\right] = 0. \tag{5.17c}$$

From the general case given by eqns (5.7) the two special cases can be obtained. The equation of the unperturbed motion (4.7) will remain the same in each case. Differences will only occur in the perturbed motion.

Axisymmetrical buckling mode. In this case the total displacements in the perturbed motion are described by the functions: $u = u_p(\phi, t)$, $v = v_p$, $w = w_0(t) + w_p(\phi, t)$. Equations of the perturbed motion can be obtained from the general equations by putting $v_p = 0$, $\partial/\partial\theta = 0$, $m = 0$. The governing equations of the viscoplastic flow buckling reduce to the two equations obtained from eqns (5.12) and (5.15). The equations for the amplitudes $u_n(t)$, $w_n(t)$ reduce to eqns (5.17b) and (5.17c). This means that the values of n for this special and the general case are the same. This case of buckling mode was considered for $\delta = 1$ in [12].

The general case described by the functions (5.1) may be considerably simplified by assuming: $u = u_p = 0$, $v = v_p = 0$, $w = w_0(t) + w_p(\phi, \theta, t)$. In such a situation the governing equation of the viscoplastic flow buckling and the equation for the amplitude $w_{mn}(t)$ obtained from eqns (5.15) and (5.17c) have the form

$$[\bar{c}C_1L(L-2) + C_3(\bar{c}L-2)]\dot{w}_p + 3C_0\frac{\dot{w}_0}{a}(L-2)w_p - a(q_z^p - \rho h\ddot{w}_p) = 0, \quad (5.18)$$

$$\ddot{w}_{mn} - \frac{1}{a\rho h}\left[(\bar{c}(\lambda_n+2)(C_3 - \lambda_n C_1) + 2C_3)\dot{w}_{mn} + 3(\lambda_n+2)\frac{\dot{w}_0}{a}C_0 w_{mn} + aq^z_{mn}\right] = 0 \quad (5.19)$$

where w_p and q_z^p are given by (5.16c).

Returning to the fundamental equations of the amplitudes, we see that the set (5.17) is separated into two subsets. The coefficients of these equations are functions of time and are determined by the solution for the unperturbed motion. However, this solution is given in the parametric form and it turns out to be impossible to obtain the effective analytical solution of eqns (5.17). The solution of these equations for given initial conditions can be obtained by numerical integration.

Bearing in mind the form of the solution (5.16), the initial conditions for the perturbations must also be expressed in series.

Let us consider a shell loaded by an impulse pressure uniformly applied to the surface and directed radially and inwards. Thus, practically, unavoidable perturbations are represented by $q_z^p \neq 0$, $q_\phi^p = q_\theta^p = 0$, $(q^\phi_{mn} = q^\theta_{mn} = 0)$ and the following sets of initial values for eqns (5.17) may be assumed:

For loading by the rectangular impulse

$$\begin{aligned} &u_{mn}(0) = 0, \quad v_{mn}(0) = 0, \quad w_{mn}(0) = a_{mn} = \bar{a}h,\\ &\dot{u}_{mn}(0) = 0, \quad \dot{v}_{mn}(0) = 0, \quad \dot{w}_{mn}(0) = 0. \end{aligned} \quad (5.20)$$

For loading by the ideal impulse, $(q_z^p = 0)$

$$\begin{aligned} &u_{mn}(0) = 0, \quad v_{mn}(0) = 0, \quad w_{mn}(0) = a_{mn} = \bar{a}h,\\ &\dot{u}_{mn}(0) = 0, \quad \dot{v}_{mn}(0) = 0, \quad \dot{w}_{mn}(0) = b_{mn} = \bar{b}V_0 \end{aligned} \quad (5.21)$$

where $\bar{a}$ and $\bar{b}$ are constants. For $q^z_{mn}(t)$ the relation $q^z_{mn} = c_{mn} = \bar{c}Q$ is assumed where $\bar{c}$ is a constant. For the initial conditions (5.20) or (5.21) we obtain from eqn (5.17a) $v_{mn} = 0$ and the functions (5.16) reduce, for the definite m and n, to the form

$$\begin{aligned} u_p &= u_{mn}(t)\frac{\mathrm{d}P_n^m(\cos\phi)}{\mathrm{d}\phi}\cos m\theta,\\ v_p &= -u_{mn}(t)\frac{m}{\sin\phi}P_n^m(\cos\phi)\sin m\theta,\\ w_p &= w_{mn}(t)P_n^m(\cos\phi)\cos m\theta. \end{aligned} \quad (5.22)$$

It is worth noting that the solution of eqns (5.17) obtained for an even n and $m \neq 1$ is also valid for the spherical half-dome, Fig. 2 with the following boundary conditions:

$$\chi_1\left(\frac{\pi}{2}, \theta, t\right) = 0, \quad u\left(\frac{\pi}{2}, \theta, t\right) = 0, \quad v\left(\frac{\pi}{2}, \theta, t\right) = 0,$$

$$Q_\phi + \frac{1}{a \sin\phi}\frac{\partial M_{\phi\theta}}{\partial\theta} - \chi_1 N_\phi - \chi_2 N_{\phi\theta}\Big|_{(\pi/2, \theta, t)} = 0 \tag{5.23}$$

where $\chi_1 = (u + \partial w/\partial\phi)/a$, $\chi_2 = [v + (\sin\phi)^{-1}\partial w/\partial\theta]/a$ denote the angles of rotations of the tangents to the meridional and circumferential circles and N_ϕ, $N_{\phi\theta}$, $M_{\phi\theta}$ and Q_ϕ are given by (5.5) and by the fifth equation of (2.4), respectively.

6. NUMERICAL RESULTS AND DISCUSSION

The equation of unperturbed motion and the equations of amplitudes have been solved numerically employing the Merson procedure. The spherical shell was loaded uniformly by an ideal pressure impulse and kept at a given temperature. The shell thickness was assumed to be 3 mm and its radius 100 mm. The shell was made of mild steel 1015 with the following material data: $\sigma_0 = 206.9$ MPa, $\gamma^* = \gamma/\sqrt{3} = 40.4\ \mathrm{s}^{-1}$ ($\bar\theta = 288$ K), $\delta = 5$ and $\rho = 7.78\ \mathrm{Mg/m^3}$. The results are also obtained for the linear yield law, $\delta = 1$.

Some of the numerical computations are presented diagrammatically. In Figs. 3 and 4 a substantial influence of the material constant δ and the temperature $\bar\theta$ on the magnitude of unperturbed radial displacement is shown. In Figs. 5–8 the amplitudes of perturbed displacements w_{mn}, u_{mn} are given as functions of t, δ, $\bar\theta$, n and V_0. The perturbation coefficients were assumed as constants, $\bar a = \bar b = 0.01$. Again, a pronounced influence of these parameters is readily observed. The solution of eqns (5.17) is indicated by solid line, the solution of eqn (5.19), accounting for the radial displacement w_{mn} only, by broken line. In Fig. 9 the shape of buckled shell is given at $t = t_f = 36.9\ \mu$s. It is seen that the u_p is many times smaller than w_p; it is the growth of normal displacement that mainly causes the instability of the shell. In Fig. 8 the variation of the $w_{mn}(t_f)$ is shown as a function of the impulse applied, V_0. The numbers at the dots distributed along the curves denote the critical modes. The function $w_{mn}(t_f)$ is monotonously increasing and the absence of

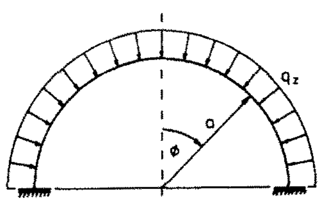

Fig. 2.

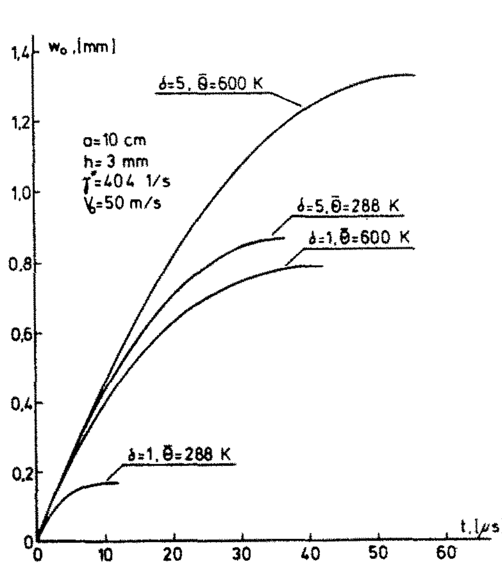

Fig. 3.

Fig. 2. Spherical half-dome, boundary conditions.

Fig. 3. Unperturbed displacements w_0 vs time.

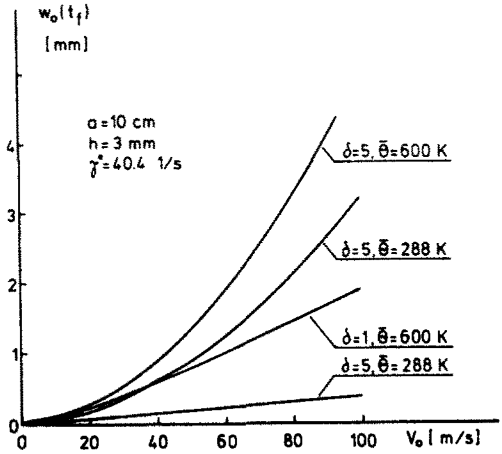

Fig. 4. Unperturbed final displacements $w_0(t_f)$ vs impulse V_0.

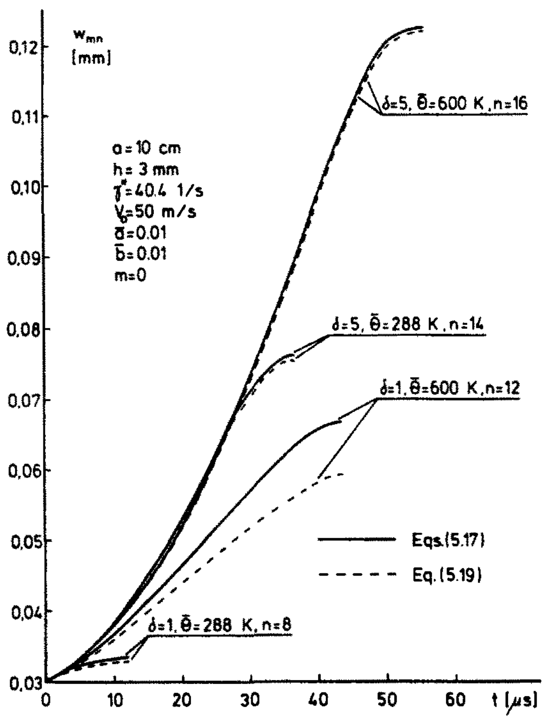

Fig. 5. Time variation of amplitudes of displacement w_{mn}.

extremum makes it impossible to accurately determine the critical impulse. In this type of buckling the loss of stability is not quite instantaneous, the process needs the increment of loading and some time to develop. The maximum amplitudes $w_{mn}(t_f)$ reach large values in a certain narrow interval of the impulse variation. Hence it is natural to determine the approximate threshold value of the impulse graphically as the abcissa of such a point on the curve at which a small increment of the pulse begins to produce considerable increments of the deflection amplitude.

In the analysis the elastic strains are neglected. The amplitudes of perturbed viscoplastic flow are restricted to be small compared with the radial displacement so the unloading does not occur, $\dot{I}_2 > 0$. Accounting for the meridional displacement, Figs. 5 and 6 leads to an increase in the radial displacement. This increase is small and is found to diminish the critical impulse.

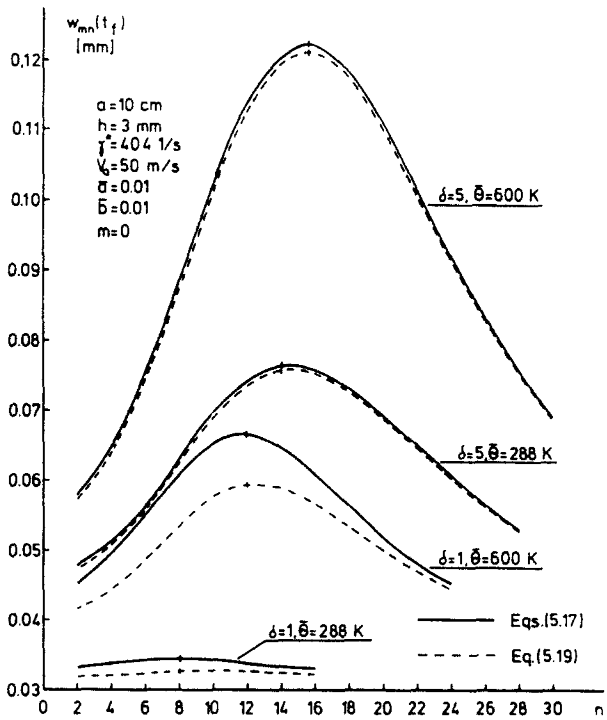

Fig. 6. Amplitudes of final perturbed displacement $w_{mn}(t_f)$ vs number of half-waves n.

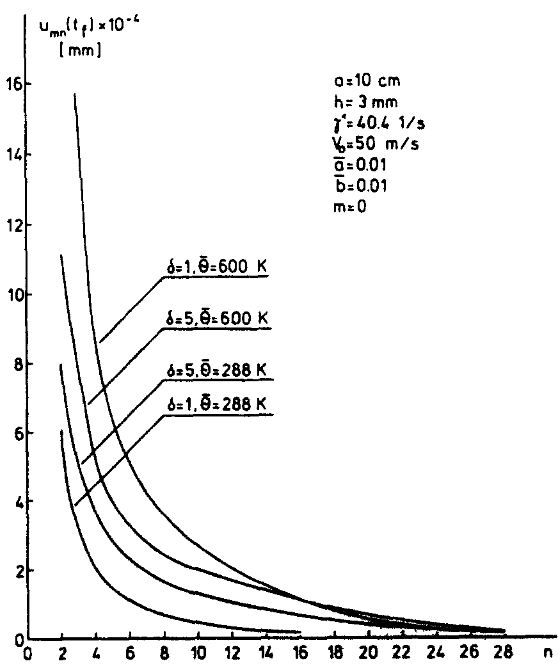

Fig. 7. Amplitudes of final perturbed displacement $u_{mn}(t_f)$ vs number of half-waves n.

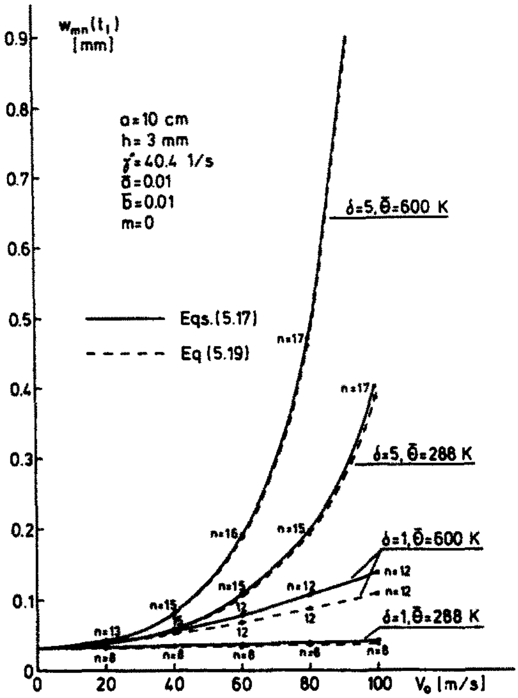

Fig. 8. Maximum amplitudes of perturbed displacement $w_{mn}(t_f)$ and the mode numbers n vs applied impulse V_0.

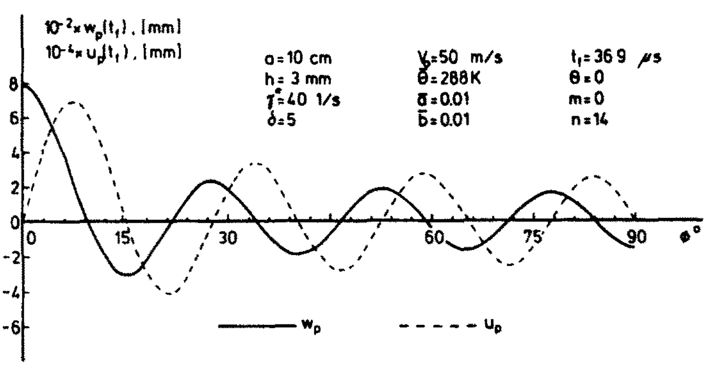

Fig. 9. Perturbed final displacements w_p and u_p.

The nonlinearity of the function $\Phi(F)$ as well as elevated temperature and the initial imperfections of the geometry and loading are the main factors which cause a considerable decrease of the buckling resistance of the shell.

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